Wave-breaking limits for relativistic electrostatic waves in a one-dimensional warm plasma

R. M. G. M. Trines and P. A. Norreys
CCLRC Rutherford Appleton Laboratory, Chilton, Didcot, OX11 0QX, United Kingdom

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The propagation of electrostatic plasma waves having relativistic phase speed and amplitude has been studied. The plasma is described as a warm, relativistic, collisionless, nonequilibrium, one-dimensional electron fluid. Wave-breaking limits for the electrostatic field are calculated for nonrelativistic initial plasma temperatures and arbitrary phase velocities, and a correspondence between wave breaking and background particle trapping has been uncovered. Particular care is given to the ultrarelativistic regime $(\gamma_\varphi^2 k_B T_0/(m_e c^2) \gg 1)$, since conflicting results for this regime have been published in the literature. It is shown here that the ultrarelativistic wave-breaking limit will reach arbitrarily large values for $\gamma_\varphi \to \infty$ and fixed initial temperature. Previous results claiming that this limit is bounded even in the limit $\gamma_\varphi \to \infty$ are shown to suffer from incorrect application of the relativistic fluid equations and higher, more realistic wave-breaking limits are appropriate.
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I. INTRODUCTION

It was shown by Dawson[1] that longitudinal wave breaking in a cold one-dimensional (1D) plasma occurs when elements of the plasma electron fluid that started out in different positions overtake each other while moving back and forth during the passage of the wave. For both nonrelativistic and relativistic plasmas, this overtaking happens when the peak fluid velocity equals the phase speed of the plasma wave.[1,2] A direct consequence of this is that a large fraction of the plasma electrons get trapped in and accelerated by the plasma wave.

Adapting this definition for use with a warm plasma is by no means straightforward. Although it is generally accepted that wave breaking implies the trapping of background plasma electrons,[3] it has been observed by several authors that electron trapping in itself does not imply wave breaking.[4,5] The reason for this is that in a thermal plasma there is always a small fraction of the electron population attaining speeds larger than the phase speed of the plasma wave under consideration. Such electrons will, of course, get trapped sooner or later, so if wave breaking were equated to background electron trapping, every longitudinal plasma wave, regardless of phase speed or amplitude, would always be broken.

In practice, plasma waves can support small numbers of trapped electrons without losing their wave structure. This is well visualized, for example, by Bergmann and Mulser,[5] as well as in simulations[6–12] and experiments[13–19] of self-injected monoenergetic electron bunches in the "bubble regime." In their paper, Bergmann and Mulser present the results of 1D Vlasov simulations showing that when a small fraction of the plasma electrons gets trapped, the periodic wave structure is unaffected, while in the case that a large fraction gets trapped, the periodic structure collapses and disappears. In this context, wave breaking is defined as "the loss of periodicity in at least one of the macroscopically observable quantities." In terms of particle trapping, this condition is satisfied if a considerable fraction of the plasma electrons are trapped, not just the fastest ones. We will refer to this approach as the "kinetic" definition for wave breaking.

In order to quantify this definition, the electron sound speed $s_0 = \sqrt{3k_B T_0/m_e}$ and associated momentum $p_0 = \sqrt{3k_B T_0 m_e}$ are introduced, where T_0 denotes the plasma temperature before the arrival of the wave. Since Langmuir waves in an electron plasma always have a phase speed larger than s_0, it follows that electrons with initial speed $|v| \le s_0$ will contribute to the collective oscillations that define the wave, while electrons with initial speed $|v| > s_0$ do not contribute to these oscillations. Longitudinal wave breaking is then defined as *the trapping (by the wave) of background plasma electrons having an initial forward momentum not larger than* p_0. In other words, a wave will break if not only electrons from the "tail" of the distribution are trapped, but also electrons originating from the "body" of the distribution. When that happens, the wave will disrupt the collective electron oscillations that drive it in the first place, and will eventually collapse and lose its structure.

In the literature, however, a number of less straightforward and often conflicting definitions for 1D wave breaking in a warm plasma have been used. All these definitions are based on the 1D Vlasov equation for electrons, to which the quasistatic approximation is applied, i.e., every quantity is assumed to be a function of $\xi = x - v_\varphi t$ only, where v_φ denotes the phase speed of the wave. Any solutions to the quasistatic Vlasov equation will break down as soon as the quasistatic assumption is violated, i.e., as soon as large-scale particle trapping starts. In that light it makes sense to equate the breakdown of the quasistatic Vlasov model to wave breaking. Such an approach is followed by, e.g., Coffey[3] and Katsouleas and Mori (K&M),[20,21] who equate wave breaking to the trapping of the upper boundary of a waterbag distribution, of which the momentum boundaries are set at $\pm p_0$ initially. It will be shown below that the above kinetic definition for wave breaking can be applied to these waterbag models, even though they are mostly hydrodynamic in nature.

In contrast, most papers on wave breaking simplify the Vlasov model to a quasistatic warm-fluid model and equate wave breaking to the breakdown of that fluid model. We will refer to this approach as the "hydrodynamic" definition for wave breaking. This works as follows. In most, if not all, fluid models for quasistatic wave propagation, the wave motion is described by the evolution of the electrostatic potential Ψ; other quantities like the mean fluid speed v and the density n are calculated from Ψ. Of course, this approach only works as long as there is an invertible relation between Ψ and v or n; as soon as $d\Psi(v)/dv=0$ or $d\Psi(n)/dn=0$, the fluid model breaks down. Unfortunately, many different warm-fluid models can be found in the literature,[22–29] leading to a range of conflicting "definitions" for wave breaking. As will be shown below, this is particularly the case for ultrarelativistic plasma waves, i.e., plasma waves obeying $\gamma_\varphi^2\beta \gg 1$, where γ_φ is the Lorentz factor corresponding to the phase speed of the plasma wave, $\beta=3k_BT_0/(m_ec^2)$, T_0 the initial plasma temperature, and m_e the electron mass. One of the aims of this paper is to create some order in this diverse landscape of warm-fluid models and establish which model is most suitable for the study of wave breaking in thermal plasma.

It is demonstrated here that great care must be taken in verifying that the singularity of any warm-fluid model used for the hydrodynamic definition for wave breaking is applied judiciously so that its physical meaning is retained. Similar care must be taken to ensure that the relativistic fluid equations are applied to the problem correctly. It is shown here that for models meeting these two criteria[20] higher, more realistic wave-breaking limits are found than for models that fail to meet them.[24–26] In an extension to the model of K&M,[20] a lower bound to the wave-breaking field is derived here for the first time, as a complement to the upper bound presented there. Using this new lower bound, it is shown that a wave having phase speed $v_\varphi=c$ in a plasma with a finite initial temperature will never break, regardless of its amplitude.

This result has important repercussions for the research into multi-GeV electron acceleration in laser or plasma wakefields.[6–19] Very large electric fields are needed for such acceleration schemes, so it would be problematic if there would exist a finite upper bound for the wave-breaking limit in a warm plasma even for $v_\varphi \to c$, as predicted by several authors.[24–26] Fortunately, it is found here that the wave-breaking limits calculated in those papers are inherently too low, and that the wave-breaking threshold in the limit $v_\varphi \to c$ tends to infinity, as derived by K&M.[20] Even though the blowout regime of Pukhov and Meyer-ter-Vehn[6] and others is essentially multidimensional in nature and reminds one of transverse wave breaking as described by Bulanov *et al.*,[30] it is important to establish the physics in one dimension first, before one can embark successfully on multidimensional models.

II. RELATIVISTIC FLUID DYNAMICS

Before embarking on the study of wave breaking, a short summary of 1D relativistic fluid dynamics[31–33] is given, since there are important differences with nonrelativistic fluid dynamics. In addition, the differences between fully relativistic fluid dynamics and the so-called warm-plasma approximation[34–36] (which is used when the mean plasma velocity is relativistic, but the thermal velocity spread is not) are clarified.

In nonrelativistic fluid dynamics, the internal energy U is a second-order moment of the velocity distribution which can be separated exactly into the mean-flow energy and the thermal energy. The energy flux is a third-order moment which can be separated exactly into the flux of mean energy, the flux of thermal energy along the mean flow, and the flux of thermal energy from one fluid element to the next. The latter quantity is often denoted as the heat flow q. The fluid pressure P is a second-order moment satisfying $P=2U$ for a 1D fluid, or $P=2U/3$ for a three–dimensional (3D) fluid. The ratio of heat capacities is given by $c_p/c_v=1+dP/dU$, so $c_p/c_v=3$ (5/3) for a 1D (3D) fluid. On adiabatic compression, i.e., when the heat flow during compression is negligible, we have $P\sim n^{c_p/c_v}$, or $P\sim n^3$ for a 1D fluid.

In relativistic fluid dynamics,[31–33] things are quite different however. The internal energy is still a second-order moment, but it cannot be separated directly into mean and thermal energy anymore, since the rules for adding velocities have changed. The relativistic energy flux is a second-order moment of the relativistic momentum distribution rather than a third-order moment, and once again it is not straightforward to separate it into mean energy transport, thermal energy transport, and heat flow. Counterintuitive as it may seem, the heat flow itself is no longer a third-order moment, but is expressed in first- and second-order moments instead.[31] The fluid pressure P is still a second-order moment, but now satisfies Taub's fundamental inequality,[32] adapted for one dimension,

$$U \geq \frac{P}{2} + \sqrt{1+(P/2)^2} - 1, \tag{1}$$

where U and P are scaled with m_ec^2. For very small or very large temperatures, both sides of the inequality are approximately equal: $U\approx P/2$ for $k_BT/(m_ec^2)\ll 1$ and $1+U\approx P$ for $k_BT/(m_ec^2)\gg 1$. It then follows that $c_p/c_v=1+dP/dU$ ranges from 3 for a nonrelativistic plasma $(U,P\ll 1)$ to 2 for an ultrarelativistic plasma $(U,P\gg 1)$. Correspondingly, the fluid pressure for an adiabatic process ranges from $P\sim n^3$ for a nonrelativistic plasma to $P\sim n^2$ for an ultrarelativistic plasma. Note that this behavior of P is fully caused by relativistic effects, independent of the actual shape of the velocity distribution of the fluid.

An approximation that is used quite often in the theory of relativistic plasma dynamics is the so-called warm-plasma approximation.[34–36] In this approximation, it is assumed that the plasma temperature is much smaller than the kinetic energy associated with the mean flow. Among other things, this allows the approximate (not exact) separation of the internal energy and energy flux into contributions comparable to the nonrelativistic case. The heat flow can then be *approximated* by third-order centered moments (centered with respect to the mean momentum), and the pressure can be written as $P\sim(n/\gamma)^3$ for not-too-extreme adiabatic compression. As such, the warm-plasma approximation is more like nonrelativistic fluid dynamics, avoiding most of the intricacies of the fully relativistic theory. As shown below, this approximation is not suitable for the study of breaking of (ultra)relativistic plasma waves, since such waves drive the plasma to temperatures for which the warm-plasma approximation is no longer applicable.

A special note of warning should be issued concerning the study of adiabatic processes in the plasma, since fast wave propagation is normally considered to be adiabatic. For nonrelativistic plasma, and in the warm-plasma approximation,[34–36] the heat flow is (approximately) given by third-order moments of the momentum distribution. Thus, for an adiabatic process, the third-order moments will vanish, providing a neat way to close the system of moment equations and arrive at a warm-fluid model. For a fully relativistic plasma, however, the heat flow is not given by the third-order moments any longer[31–33] and setting the third-order moments to zero is no longer equivalent to considering an adiabatic process. In fact, the third-order moments can grow fairly large for ultrarelativistic adiabatic processes [$\sim p^3\log(p)\gg1$, where $p\gg1$ is the average fluid momentum] and uncompromisingly forcing them to zero will lead to incorrect results, possibly even violation of the first and second law of thermodynamics.

III. HYDRODYNAMIC APPROACH TO WAVE BREAKING

In this section, the various 1D warm-fluid models that have been used to determine wave-breaking limits for a warm plasma are examined. We will verify the extent to which these models obey the fundamental inequality (1) and use this as a criterion to determine which model is most suitable to study wave breaking; in particular, in the regime $\gamma_\varphi^2\beta\gg1$.

For convenience, the following scalings are applied: $t\mapsto\omega_p t$, $x\mapsto\omega_p x/c$, $v\mapsto v/c$, $p\mapsto p/(m_e c)$, $U\mapsto U/(m_e c^2)$, $P\mapsto P/(m_e c^2)$, $E\mapsto E/(m_e\omega_p c)$, $n\mapsto n/n_0$, where m_e denotes the electron mass, n_0 the plasma background density, and $\omega_p=e^2 n_0/(\varepsilon_0 m_e)$ the plasma frequency.

The 1D relativistic Vlasov equation for the distribution function $f(t,x,p)$ is given by

$$\frac{\partial f}{\partial t}+\frac{p}{\gamma}\frac{\partial f}{\partial x}-E\frac{\partial f}{\partial p}=0. \tag{2}$$

Taking the various moments of this equation provides a system of equations of which the moment functions $M_n\equiv\int p^n f(p)dp$ are to be the solutions. Truncating the system of moment equations at some point allows one to calculate the moments without actually knowing the precise solution to Eq. (2) that they are supposed to correspond to. The danger in this approach lies in the fact that there is no built-in guarantee that the moment solutions found this way will correspond to the moments of an actual solution to Eq. (2). Nevertheless, a judicious truncation of the system of moment equations should ensure that there is a solution to Eq. (2) having a set of moments that lie very close to the moments resulting from the truncated system. An ill-advised truncation, however, can break the connection between the original Vlasov solution and the moment solutions, causing the moment solutions to behave differently from the Vlasov solution they are supposed to represent. In particular, they might break down under different conditions than the (quasistatic) Vlasov solution; if such breakdown is taken to represent wave breaking, the moment solutions will yield an incorrect wave-breaking threshold. While it is often argued that breakdown of a warm-fluid model is equivalent to wave breaking,[22,24–26,28,29] it is shown in this section and the next that things are not that simple: if the breakdown of a warm-fluid model is not matched by a breakdown of the underlying quasistatic Vlasov solution, then such breakdown is a mere mathematical feature of the warm-fluid model without any physical relevance.

Warm-fluid models are usually obtained by truncating the system of moment equations beyond the second-order equation (the energy equation). Closure is then obtained in either one of two ways. First, one can make an educated guess concerning the shape of the distribution function, and use that to express the second-order moments in terms of the density and the first-order moments.[3,20] Second, one can assume that the third-order moments are negligibly small, drop them from the system of moment equations, and work back from there to solve for the lower-order moments; this is the so-called *warm-plasma approximation*.[34–36] Note that the warm-plasma approximation is basically a low-density/low-temperature expansion, and not, as stated erroneously in Ref. 26, a representation of an adiabatic process. Either way, one ends up with expressions for the internal energy U and pressure P in terms of the plasma density, temperature, and mean momentum. These expressions provide one with a handle to distinguish between various models: to study breaking of ultrarelativistic waves, we need a model that satisfies Taub's fundamental inequality (1) even for ultrarelativistic pressures $P\gg1$, and not every model satisfies this condition.

In the model of K&M,[20] we find the following expressions for U and P in a fluid element's rest frame:

$$\begin{Bmatrix} P \\ U \end{Bmatrix}=\frac{1}{2p_0}\left[n_p p_0\sqrt{1+(n_p p_0)^2}\mp\sinh^{-1}(n_p p_0)\right],$$

where $n_p=n/\gamma$, $\gamma=1/\sqrt{1-v^2}$ is the Lorentz factor associated with the mean flow, and $p_0=\beta^{1/2}$, where $\beta=3k_B T_0/(m_e c^2)$ as before. It can easily be verified that these expressions satisfy the inequality (1) for arbitrary values of $n_p p_0$, rendering this model suitable for the study of wave breaking in both limits $\gamma_\varphi^2\beta\ll1$ and $\gamma_\varphi^2\beta\gg1$.

Next, three models[24–26] based on various warm-plasma approximations[34–36] are investigated. The (adiabatic) pressure terms used in these models all behave like $P\sim n^3$ for all n, while showing subtle differences: Reference 24 uses $P=(\beta/3)(n/\gamma)^3$, where γ denotes the Lorentz factor corresponding to the mean flow speed v; Ref. 25 uses $P=(\beta/3)n^3/\gamma$ (and additional terms in the advection equation to account for the mass increase due to thermal motion); while Ref. 26 uses $P=(\beta/3)(n/\gamma_{th})^3/\gamma$, where γ_{th} denotes the Lorentz factor due to the mean thermal fluctuations in the plasma (eventually set to 1, thus neglecting the "thermal" mass increase). Although these papers do not provide expressions for U, it is nevertheless possible to test them against the inequality (1), as this equation predicts that, for an adiabatic process, $P\sim n^3$ for small n, T while $P\sim n^2$ for large n, T. One immediately sees that, while these models satisfy (1) in the low-density, low-temperature limit, they do not do so in the high-density, high-temperature limit, even though all these models claim that their expressions for P correspond to adiabatic compression. This is of course in line with the warm-plasma approximations being low-density, low-temperature approximations that cease to be valid for very high compression ($P\gg 1$). Since the study of wave breaking involves large-amplitude waves that may severely compress the plasma, it follows that the models of Refs. 24–26 are not really suitable for the study of wave breaking, even though they appear to be more "general" than the waterbag model of K&M. This will be expanded on in Sec. VII.

It must be stressed that the differences between the models of K&M and the others stem exclusively from the different equations of state used by these models (resulting from the more rigorous treatment of relativistic effects by K&M), and that equations of state similar to those of K&M can also be found for distribution functions different from a waterbag. Contrary to a statement in Ref. 24, the differences between the various models do not arise from algebraic errors in the work of K&M. The claims made in Ref. 25, that these differences are caused by K&M using a kinetic definition for wave breaking rather than a hydrodynamic one, are equally unfounded: as remarked by K&M (and verified below), their model yields identical results regardless of which definition of wave breaking is used. The methods used in Ref. 25 to obtain a wave-breaking threshold from a given fluid model are mathematically equivalent to the hydrodynamic definition given in Sec. I; applying these methods to K&M's model yields K&M's results, not those of Ref. 25, refuting any claims to the opposite.

It has also been claimed[25] that the wave-breaking limit presented in Ref. 24 is in better agreement with the results of certain Vlasov simulations by Krall *et al.*[37] However, upon closer scrutiny of Ref. 37, it turns out that a wave having $\gamma_\varphi=100$ is used in the simulations presented there, while it is assumed that particle trapping occurs as soon as a particle leaves the simulation space, which is defined by $-10.2\leq p_z \leq 30.7$. Of course, for $\gamma_\varphi=100$, a particle having $p_z=30.7$ is still a long way from being trapped. Also, wave breaking is defined in Ref. 37 by the trapping of the fastest particles in the distribution, even though trapping of a small number of very fast particles is not enough to trigger wave breaking, as pointed out in Sec. I and Ref. 5. Indeed, the waves pictured in Ref. 37 look very regular and unbroken. It must therefore be concluded that these simulations do not show any evidence for wave breaking. Since the electric-field amplitudes of the plasma waves found in these simulations easily reach the wave-breaking limit according to Refs. 24–26, while the waves themselves are evidently not broken yet, these results support the notion that the wave-breaking limit according to Refs. 24–26 is too low, rather than K&M's values being too high, refuting claims to the opposite.[25,37]

In Ref. 26, it is claimed that the models by K&M[20] and Rosenzweig[24] are similar and different from the model of Ref. 26, since the results of the first two are only given in the limit $\gamma_\varphi^2\beta\gg 1$, while the third also gives the wave-breaking limit in the regime $\gamma_\varphi^2\beta\ll 1$. These claims are not valid for the following reasons: First, the models of Refs. 20 and 24 can easily be evaluated for the regime $\gamma_\varphi^2\beta\ll 1$ also, even if the respective authors did not happen to do so; and second, the equation of state in both Refs. 24 and 26 reads $P\sim(n/\gamma)^3$, while K&M use $P\sim(n/\gamma)\sqrt{1+(n/\gamma)^2}$, rendering the models of Refs. 24 and 26 similar and different from that of K&M.[20]

There exist several lesser-known models for wave breaking in plasmas, such as the three-fluid model[27] or a model based on the method of characteristics.[28,29] These models contain both kinetic and hydrodynamic elements and are not so easily classified; they will be discussed in detail in Sec. VI.

IV. KINETIC APPROACH TO WAVE BREAKING

As stated in Sec. I, wave breaking and background particle trapping are intimately related, and many attempts were made in the past to unite them. However, such attempts were often complicated by the fact that most models for plasma wave propagation employ the quasistatic approximation, which does not tolerate any particle trapping at all: as soon as the first particle gets trapped, any quasistatic model will break down immediately. The solution to this is to employ a quasistatic model based on a so-called waterbag distribution:[3,38] a distribution that is a nonzero constant for thermal speeds smaller than the electron sound speed s_0, and zero otherwise. This distribution behaves like a Gaussian distribution in many aspects, but particle trapping will be postponed until particles at (initially) the electron sound speed (i.e., at the upper bag boundary) can be trapped. Thus, a quasistatic waterbag model will break down at the same instant that large-scale particle trapping sets in for a non-quasistatic Gaussian model, triggering wave breaking according to our original kinetic definition even though the waterbag is not a fully kinetic model in itself. This renders the quasistatic waterbag model quite suitable for the study of wave breaking, since the point of breakdown can be determined analytically to great accuracy for this model.

The quasistatic waterbag model has been employed explicitly by Coffey[3] and K&M,[20] all of whom define wave breaking by the trapping of the upper waterbag boundary. It should be noted however, that most other work on wave breaking still favors the waterbag distribution implicitly, even if it is not mentioned. This is because almost all work on plasma wave breaking starts from the following assumptions: first, the wave is quasistatic, i.e., the effect of small amounts of trapped particles on an unbroken wave is neglected, and second, its propagation is an adiabatic process. As it happens, the waterbag distribution is the only one that satisfies both assumptions exactly, so it is implicitly favored over other distributions, which can only satisfy the assumptions approximately. So even papers that argue that they do not restrict themselves to a particular distribution function are still doing so, and are thus less general than they claim to be.

We have studied particle trapping for a wave on the verge of breaking for two different models: the fully relativistic model of K&M,[20] and the weakly relativistic model of Refs. 24–26 For this, we used the Hamiltonian approach of Ruth and Chao for particle dynamics in a wakefield,[39] where the wakefield potential was of course taken from the various plasma models we considered. It should be noted that for proper application of this method, a few common mistakes made by several authors (see Ref. 24 and in particular, Ref. 40) should be avoided: (i) one cannot express a particle's velocity directly as the sum of a mean velocity and a thermal part, since addition of relativistic velocities is governed by special rules; (ii) describing a warm plasma as a distribution of test electrons governed by the wakefield potential of a cold plasma is not correct, as it will miss out the effect of pressure on the relationship between mean velocity and electrostatic fields; (iii) the average speed of the electron fluid is not equal to the speed of a particle that is initially at rest; in fact, the mean fluid flow does not correspond to any individual particle orbit; and (iv) when the wakefield potential for a warm plasma is used, one must not add an additional term for the plasma pressure to the Hamiltonian, since the plasma pressure does not act on individual electrons. (See the Appendix for details.)

To proceed, in the Hamiltonian approach[39] the (quasistatic) Hamiltonian of a particle in a plasma wave is given by

$$H = \gamma_p(1 - v_\varphi v_p) - \Psi(v), \tag{3}$$

where v_p and γ_p denote the velocity and Lorentz factor of the particle, and $\Psi(v)$ the wakefield potential as a function of the mean plasma flow speed v. Particles in the plasma wave will follow curves of constant H. Before the arrival of the plasma wave, a particle at the upper boundary of a waterbag distribution has $p_p=p_0=\sqrt{\beta}$ and $H=H_0=\sqrt{1+\beta}-v_\varphi\sqrt{\beta}-\Psi(0)$. When this particle gets trapped, we have $p_p=\gamma_\varphi v_\varphi$ and $H(v)=1/\gamma_\varphi-\Psi(v)$. The mean flow at the instant of trapping can be determined by solving $H(v)=H_0$ or

$$\Psi(v) - \Psi(0) = 1/\gamma_\varphi - \sqrt{1+\beta} + v_\varphi\sqrt{\beta}. \tag{4}$$

For the waterbag model by K&M,[20] the wakefield potential is given by

$$\Psi_{KM}(v) = \frac{1 - v v_\varphi}{(1-v^2)^{1/2}}\left[1 + \beta\frac{1-v^2}{(1-v/v_\varphi)^2}\right]^{1/2}. \tag{5}$$

Inserting this expression into (4), solving for v, and taking the smallest solution, we find the minimal flow v_0 needed to trap particles initially at $p_p=p_0$ (i.e., the upper boundary of the waterbag),

$$v_0 = v_\varphi\frac{1 - \sqrt{\beta/\gamma_\varphi^2 + \sqrt{\beta}/(\gamma_\varphi^3 v_\varphi)}}{1 + v_\varphi\sqrt{\beta}/\gamma_\varphi}. \tag{6}$$

The critical speed v_0 at which this model will break down is found by solving $\partial\Psi_{KM}(v)/\partial v=0$ for v, and taking the smallest solution; this procedure yields the exact same value as that given in Eq. (6), for all values of γ_φ or β. This proves that for the model of K&M[20] the kinetic and hydrodynamic definitions of wave breaking coincide and validates the use of this model to calculate wave-breaking thresholds in all regimes.

For the other models,[24–26] things are different. For these models, the potential $\Psi(v)$ is given by (with small variations)

$$\Psi(v) = \frac{1 - v v_\varphi}{(1-v^2)^{1/2}} + \frac{\beta}{2}\frac{(1 - v v_\varphi)(1-v^2)^{1/2}}{(1 - v/v_\varphi)^2}.$$

This potential can immediately be recognized as a low-temperature expansion of (5), and it is safe to assume that in the regime $\gamma_\varphi^2\beta \ll 1$, it will not behave very differently from $\Psi_{KM}(v)$. We will therefore concentrate on the limit $\gamma_\varphi\to\infty$ while keeping β fixed, as this is where the differences between the models are most visible. Defining $\chi^2=(1-v)/(1+v)$, one finds that $\Psi(v)=\chi+\beta/(2\chi)$ in this limit, and Eq. (4) reduces to $\chi+\beta/(2\chi)=1+\beta/2-\sqrt{1+\beta}+\sqrt{\beta}$. However, this equation does not even have a real-valued solution for χ when $\beta<4.36$, which covers the entire domain of validity of these models ($\beta\ll 1$). This shows that any wave described by these models will not be able to trap particles having initial momentum p_0. According to this model, the slowest particles that can be trapped by a wave with amplitude $\chi_{\min}=\sqrt{\beta/2}$ (corresponding to the largest amplitude allowed before the model breaks down) still have a speed $v_{\min}\approx\sqrt{2\beta}>s_0$. One finds that for this model, one cannot make the kinetic and hydrodynamic definitions for wave breaking coincide because the wave amplitude at which the model breaks down is too low to allow for the trapping of particles having initial speed s_0. This renders the model of Refs. 24–26 unsuitable for the study of wave breaking in the regime $\gamma_\varphi^2\beta\gg 1$, from a kinetic as well as from a hydrodynamic point of view.

We conclude that for the model of K&M,[20] it is possible to reconcile wave breaking and particle trapping: hydrodynamic wave breaking coincides *exactly* with the trapping of particles that have an initial momentum of *exactly* $\sqrt{\beta}$. Using the same methods, it is possible to prove the same for the nonrelativistic waterbag model of Coffey.[3] In a recent paper,[40] a separate attempt was made to find a correspondence between wave breaking and particle trapping; unfortunately, this paper used the wakefield potential of a *cold* plasma in its single-particle Hamiltonian, even though it was aiming to study particle dynamics in a *warm* plasma. (See, also, the Appendix.) For that reason, only an approximate correspondence was found (which only holds for $\gamma_\varphi^2\beta\ll 1$), using particles that have an initial momentum of approximately $0.87\cdot\sqrt{\beta}$, while the exact correspondence that exists for certain models[3,20] was not found in that work.

V. WAVE-BREAKING LIMIT FOR RELATIVISTIC WAVES IN A WARM PLASMA

In this section, both upper and lower bounds for the wave-breaking limit for the electric field of a relativistic wave in a warm plasma are derived. These limits are obtained from the model by K&M,[20] as it has been shown above that this is the only model that can be trusted in the ultrarelativistic regime. The model itself is not derived here; only the resulting equations are presented.

Starting from a warm plasma with initial density n_0 and temperature T_0, and assuming that all quantities are functions of $\xi = x - v_\varphi t$ only, the equations of K&M read

$$\frac{\partial \Psi(v(\xi))}{\partial \xi} = -E, \quad \frac{\partial E}{\partial \xi} = 1 - n, \quad n = \frac{v_\varphi}{v_\varphi - v}, \tag{7}$$

where v is the mean plasma velocity, E the electrostatic field (scaled by $m_e \omega_p c/e$), n the plasma density (scaled by n_0), and v_φ the phase speed of the wave, while the fully relativistic wakefield potential $\Psi(v)$ for a warm plasma is given by Eq. (5) above. Combining these equations yields the energy equation

$$\frac{\partial}{\partial \xi}\left[\frac{1}{2}E^2 + V(v(\xi))\right] = 0, \quad \frac{dV(v)}{dv} = -\frac{v}{v_\varphi - v}\frac{d\Psi(v)}{dv}. \tag{8}$$

At $v = v_0$, we have $dV/dv = d\Psi/dv = 0$, while $dV/d\Psi \neq 0$. For $v > v_0$, $V(v)$ is not meaningful and $V(\Psi)$ does not exist, since $\Psi(v)$ is only invertible for $v \leq v_0$.

The wave-breaking field E_{wb} is, by definition, the field amplitude of a wave on the verge of breaking. This field can be calculated by applying Eq. (8) to a wave having the critical velocity v_0 given by Eq. (6) for its peak velocity. However, finding the wave-breaking field is not straightforward for the wakefield potential given by Eq. (5), as it is not possible to find an exact analytical expression for the function $V(v)$. This problem can be solved by using an approximation for $\Psi(v)$ to determine an approximation for $V(v)$, while still using $\Psi(v)$ itself to determine the critical speed v_0 (Ref. 20). A good approximation can be obtained by replacing $\Psi(v)$ by its limit for $v_\varphi \to 1$, which is denoted by $\Psi_1(v)$,

$$\Psi_1(v) = \sqrt{\chi^2 + \beta}, \quad \chi^2 \equiv (1 - v)/(1 + v),$$

with corresponding function $V_1(v)$,

$$V_1(v) = \frac{1}{2\sqrt{\beta}} \ln\left[\frac{\sqrt{\beta}\sqrt{\chi^2 + \beta} + \beta}{\chi}\right] + \frac{1}{2}\sqrt{\chi^2 + \beta}.$$

From the fact that $\Psi_1'(v) < \Psi'(v)$ for all $|v| < v_\varphi$, it can be derived that $V_1(v_0) > V(v_0)$, which implies that using V_1 rather than V will provide an *upper bound* for the wave-breaking limit E_{wb}. In addition to this upper bound, lower bounds for the wave-breaking field are provided here, for the "laser-wakefield" regime $\gamma_\varphi^2 \beta \ll 1$ as well as for the ultrarelativistic regime $\gamma_\varphi^2 \beta \gg 1$.

In the laser-wakefield regime, it is found that $v_0 \approx v_\varphi(1 - \beta^{1/4}/\gamma_\varphi^{3/2})$, so $\chi_0^2 \approx (1 + 2\gamma_\varphi^{1/2}\beta^{1/4})/(4\gamma_\varphi^2)$. Using the function $V_1(v)$, the following upper bound for the wave-breaking field is obtained:

$$\frac{1}{2}E_{wb}^2 \lesssim \gamma_\varphi - 1 - \gamma_\varphi(\gamma_\varphi^{1/2}\beta^{1/4} - \gamma_\varphi \beta^{1/2}). \tag{9}$$

In order to find a lower bound to the wave-breaking field, the approximated wakefield potential Ψ_2 is introduced,

$$\Psi_2(v) = \frac{1 - v v_\varphi}{(1 - v^2)^{1/2}} + \frac{\beta}{2}\frac{(1 - v v_\varphi)\sqrt{1 - v^2}}{(1 - v/v_\varphi)^2}.$$

As it happens, $\Psi_2'(v) > \Psi'(v)$ for all $|v| < v_\varphi$, guaranteeing that application of Ψ_2 will provide a lower bound for E_{wb}. In addition, Ψ_2 will provide a good approximation for Ψ in the laser-wakefield regime, where temperature effects can be seen as a small perturbation of the cold-plasma case. Repeating the above procedure for Ψ_2 rather than Ψ_1, the following lower bound for the wave-breaking field is found:

$$\frac{1}{2}E_{wb}^2 \gtrsim \gamma_\varphi - 1 - \gamma_\varphi\left(\frac{8}{3}\gamma_\varphi^{1/2}\beta^{1/4} - 2\gamma_\varphi \beta^{1/2}\right). \tag{10}$$

These results can readily be recognized as the cold relativistic limit derived by Akhiezer and Polovin,[2] with small thermal corrections. Note that both results are almost identical to the result obtained by Schroeder *et al.*,[26] for this regime. Similar results can also be obtained by inserting the value of χ_0 derived above into the models of other authors[24,25,27,29] This is not surprising, since the differences that exist between the various models used to describe a relativistic thermal plasma are not yet significant for $\gamma_\varphi^2 \beta \ll 1$.

Things are different in the ultrarelativistic regime ($\gamma_\varphi^2 \beta \gg 1$). While thermal effects could be treated as a small correction to a cold-fluid model before, they will become a dominant force in this regime. For $\gamma_\varphi^2 \beta \gg 1$, $v_0 \approx v_\varphi(1 - 2\sqrt{\beta}/\gamma_\varphi)$, so $\chi_0^2 \approx \sqrt{\beta}/\gamma_\varphi$ and the upper limit is given by

$$\frac{1}{2}E_{wb}^2 \lesssim \frac{1}{2\sqrt{\beta}} \ln(2\gamma_\varphi^{1/2}\beta^{1/4}). \tag{11}$$

For the lower limit in the ultrarelativistic regime, Ψ_2 is not a good approximation for Ψ anymore, in particular for $v > v_\varphi(1 - \beta v_\varphi - \beta v_\varphi^2)$. One therefore introduces the approximation $\Psi(v) \approx \Psi_3(v)$ for $v > v_\varphi(1 - \beta v_\varphi - \beta v_\varphi^2)$,

$$\Psi_3(v) = \sqrt{\beta}\frac{1 - v v_\varphi}{1 - v/v_\varphi} + \frac{1}{2\sqrt{\beta}}\frac{(1 - v v_\varphi)(1 - v/v_\varphi)}{1 - v^2}.$$

Again, $\Psi_3'(v) > \Psi'(v)$ for all $|v| < v_\varphi$, but Ψ_3 has been chosen to approximate Ψ well in the ultrarelativistic regime. Repeating the process yields the following lower bound for E_{wb}:

$$\frac{1}{2}E_{wb}^2 \gtrsim \frac{1}{2\sqrt{\beta}} \ln(\gamma_\varphi^{1/2}\beta^{1/4}). \tag{12}$$

The lower and upper limit only differ by the amount $\ln(2)/(2\sqrt{\beta})$, thus providing a reasonable estimate for E_{wb}. It also shows that the upper limit, which was first derived by K&M,[20] provides a much better estimate for E_{wb} in the ultrarelativistic regime than it is usually given credit for. Even more importantly, it can be seen that *both* the lower and the upper limit approach infinity for fixed β and $\gamma_\varphi \to \infty$. In other words, a wave having phase speed $v_\varphi = 1$ and finite amplitude cannot break at all. This is consistent with the observation that a wave having $v_\varphi = 1$ and finite amplitude must not be capable of trapping plasma electrons having thermal speeds below s_0 and accelerating these to the speed of light, i.e., infinite energy. It also clearly proves that the wave-breaking limit predicted by Refs. 24–26 is too low in the limit $v_\varphi \to 1$. The reason for this is that these models break down before the wave truly breaks, and that model breakdown is mistaken for wave breaking.

Since intense plasma waves are often driven by intense laser pulses, it is interesting to investigate the influence of the presence of a strong laser field on wave breaking. In a 1D setting, one finds that $p_\perp = a_\perp$, where $a_\perp = eA_\perp / m_e c$ denotes the normalized vector potential of the laser field, while transverse and longitudinal motion are decoupled, so the Lorentz factor of a plasma particle is given by $\gamma_\perp \sqrt{1+p_z^2}$, and for a fluid element by $\gamma_\perp / \sqrt{1-v^2}$, with $\gamma_\perp = \sqrt{1+a_\perp^2}$. The wave-breaking limits in the presence of a laser field can now readily be obtained by replacing $1/\sqrt{1-v^2}$ by $\gamma_\perp/\sqrt{1-v^2}$ in Eq. (5), and repeating the process. The expressions for E_{wb} then receive an additional factor of $\gamma_\perp^{1/2}$, while β is replaced by $\beta/\gamma_\perp^2$. For $\gamma_\varphi^2 \beta \ll 1$, it follows that $E_{wb}^2 \approx 2\gamma_\perp(\gamma_\varphi - 1) - 2\gamma_\perp \gamma_\varphi [(8/3)\gamma_\varphi^{1/2}\beta^{1/4}/\gamma_\perp^{1/2} - 2\gamma_\varphi \beta^{1/2}/\gamma_\perp]$, while for $\gamma_\varphi^2 \beta \gg 1$ it is found that $E_{wb}^2 \sim (\gamma_\perp^2/\sqrt{\beta}) \ln(\gamma_\perp^{-1/2}\gamma_\varphi^{1/2}\beta^{1/4})$. For large laser intensities the wave-breaking field increases significantly, while the influence of thermal effects decreases, in agreement with earlier findings.[26] For that reason, wave breaking usually occurs immediately after a driving laser pulse rather than in it.

Finally, in the nonrelativistic limit $v_\varphi^2 \ll 1$, the function $\Psi(v)$ given by Eq. (5) reduces to

$$\Psi_{nr}(v) = \frac{1}{2}v^2 - v_\varphi v + \frac{1}{2}\frac{\beta}{v_\varphi^2}\frac{v_\varphi^4}{(v_\varphi - v)^2},$$

which is equivalent to the model used by Coffey.[3] The resulting value for E_{wb} is therefore identical to Coffey's result, validating the model of K&M[20] in the nonrelativistic limit also.

VI. ALTERNATIVE APPROACHES TO WAVE BREAKING

Apart from the well-known models for wave breaking by K&M[20] and others,[24–26] there are several alternative models for relativistic wave breaking in warm plasma. These models incorporate both kinetic and hydrodynamic elements and it is less easy to classify them. Two of them will be discussed here.

A. Three-fluid model

The first model to be investigated is the three-fluid model of Ref. 27. In this model, the warm plasma is modeled using three cold fluids having density 1/3 each, and mean velocities v_t, $-v_t$, and 0, respectively. Closure of the system of fluid equations is obtained by imposing density conservation for each fluid component separately, rather than for the plasma as a whole. Equations for the evolution of the fluids are then derived in the limit $\gamma_\varphi \to \infty$. Wave breaking is defined by the trapping of the forward-moving fluid component by a passing plasma wave.

There are several problems with this approach. First, the effective temperature of the unperturbed fluid is given by $\beta = 3k_B T_0/(m_e c^2) = 2v_t^2$, so the effective electron sound speed is given by $s_0 = v_t\sqrt{2}$. This means that even the fastest particles in the distribution will be considerably slower than s_0, which is not a good model for a warm fluid and causes problems regarding the use of the wave-breaking definition of Sec. I. Second, the nonstandard way of obtaining closure causes the density of the forward-moving component to increase explosively near wave breaking, dominating the entire evolution of the fluid, while the densities of the other two components stay far behind. As a result, the effective pressure behaves like $P \sim n^{1/2}$ near wave breaking, which is far too low according to the inequality (1).

In all, the three-fluid model behaves more like a cold than a warm fluid. This is also reflected in the "wave-breaking limit" provided by this model. After correcting an algebraic error in the derivation of Ref. 27, the limit was found to be $E_{wb}^2 \approx \gamma_\varphi/3 + 1/\sqrt{2\beta}$, which is much like the limit for a cold relativistic fluid[2] with a small thermal correction, even in the limit $\gamma_\varphi^2 \beta \gg 1$. We conclude that this model does not provide a good description for a warm relativistic plasma, and that the wave-breaking limit it provides is not reliable.

B. Method of characteristics

Another method that has been employed by several authors[28,29] is based on the so-called method of characteristics, which can be used to obtain solutions to the quasistatic Vlasov equation for arbitrary initial velocity distributions, i.e., not just waterbags. The method of characteristics states that solutions of the quasistatic Vlasov equation propagate along the contours of the quasistatic Hamiltonian given by Eq. (3). For a given initial distribution $f_0(v)$, the distribution f is then found to be $f(v) = f_0(g(H(v)))$, where the nonevolving function $g(h)$ satisfies $g(H(v)) = v$ for $\Psi(v) \equiv 0$. It can immediately be seen that the evolution of f is fully governed by the evolution of the electrostatic potential Ψ through the evolution of H. An evolution equation for Ψ is then obtained through integration of f, using the method of steepest descent.[41] Since this method assumes that f is quite narrow around its maximum, implying a nonrelativistic plasma temperature, the resulting fluid model is only valid in the regime $\gamma_\varphi^2 \beta \ll 1$, as already observed by Khachatryan.[29]

While this approach shows much promise for the study of normal wave propagation in warm plasma, its application to wave breaking runs into several problems. First of all, the quasistatic Vlasov equation, on which the method is based, does not tolerate any particle trapping. Close scrutiny of the solutions presented in Refs. 28 and 29 reveals that this issue is resolved by silently dropping each and every particle from the distribution that reaches speed v_φ from below. This is acceptable for normal wave propagation, where the number of trapped particles will be small, but not for the study of wave breaking, when large-scale particle trapping occurs. Normally, the load of large amounts of trapped particles would cause the wave structure to collapse or the quasistatic Vlasov solution to break down, but this does not happen for the model of Refs. 28 and 29, since all trapped particles are stripped from the distribution.

A second, related problem is that the evolution equation for Ψ never breaks down, regardless of how many particles have been trapped and subsequently removed from the Vlasov distribution. As pictured in Ref. 29, the removal of trapped particles causes the plasma density to decrease with increasing compression when the plasma is compressed beyond a certain threshold. Of course, the model ceases to provide a valid description for the plasma as soon as that happens, but the respective authors of Refs. 28 and 29 seem to be unaware of this and continue to push their model far beyond its domain of validity. This makes it difficult to properly assess a number of predictions made by this model, in particular the wave-breaking limit on the wave amplitude and the general behavior of a wave near breaking, since it is precisely in that regime that the model loses its validity.

However, reliable results can be obtained from this model when it is only used up to the point where the density no longer increases for increasing compression, indicating the onset of large-scale particle trapping and, possibly, wave breaking. For example, Khachatryan[29] finds that the peak density at this point is given by $\sim 0.6 \cdot \gamma_\varphi^{3/2} v_\varphi^{1/2} / \beta^{1/4}$, which is very close to the value of $\gamma_\varphi^{3/2} v_\varphi^{1/2} / \beta^{1/4}$ yielded by the model of K&M[20] at wave breaking in the regime $\gamma_\varphi^2 \beta \ll 1$. This indicates that the model of Refs. 28 and 29 might still be used to study wave propagation and breaking in a thermal plasma with an arbitrary velocity distribution, as long as it is used with care and not pushed beyond its inherent limitations.

VII. DISCUSSION AND CONCLUSIONS

The first topic that needs to be discussed is the influence of the initial distribution on the relation between the plasma pressure and density on adiabatic compression. Until now, expressions for the plasma internal energy and pressure have been derived using a waterbag distribution, and it has been argued that such an approach is not sufficiently "general." However, it is not possible to derive an equation of state that will be exact for every possible velocity distribution; one always has to use certain assumptions or approximations, so no equation of state can be truly "general." On the other hand, the leading-order behavior of pressure versus density, $P \sim n^3$ for small adiabatic compression, $P \sim n^2$ for large compression, is dictated by the fundamental inequality (1), which is fully independent of the choice of distribution. And it is this leading-order behavior that is at the bottom of the differences between K&M's model and the others.

It has been argued[24,25] that even in the limit $\gamma_\varphi \to \infty$ the diverging plasma pressure will inhibit compression beyond a certain point, effectively putting a finite upper bound on the plasma density, induced by wave breaking, for any fixed $\beta > 0$. However, the existence of such an upper bound depends on the rate at which the pressure P in a frame moving with the plasma flow diverges. It appears that if $P \sim n_p^3$, where n_p is the plasma density in the comoving frame, the thermal energy will eventually dominate, leading to an upper bound on compression, while if $P \sim n_p^2$, the kinetic energy associated with the plasma flow will be narrowly dominant over the thermal energy for $\gamma_\varphi \to \infty$, and there will be no upper bound on compression. As has been shown above, $P \sim n_p^2$ rather than $P \sim n_p^3$ for the (very) large densities that can be expected in a wave on the verge of breaking. This means that for $\gamma_\varphi \to \infty$ and fixed β, P does not diverge sufficiently fast for $v \uparrow v_\varphi$ for the thermal energy to dominate over the flow energy, and there should thus be no wave-breaking-induced limit on compression in the limit $v_\varphi \to 1$. This is reflected in the value of the mean flow v_0, at which $d\Phi(v)/dv = 0$, and the electric-field amplitude E_{wb} that is needed to reach $d\Phi/dv = 0$. It can easily be shown that in the limit $v_\varphi \to 1$, K&M find that $v_0 \to 1$, so $n_{wb} \to \infty$ and $E_{wb} \to \infty$. Conversely, Rosenzweig and others find in the limit $v_\varphi \to 1$ that $v_0 \to (1-\beta/2)/(1+\beta/2) < 1$, causing both n_{wb} and E_{wb} to tend to finite values. Note that K&M's findings for $v_\varphi \to 1$ hold for any model obeying the inequality (1), as all such models satisfy $P \sim n^2$ for large compression, irrespective of the details of the distribution function.

All this is in agreement with the Vlasov picture of wave breaking. For finite γ_φ, i.e., $v_\varphi < 1$, wave breaking in a thermal plasma occurs when large amounts of plasma particles are pushed across the trapping separatrix because of the extreme pressure at the peak plasma density. However, for $v_\varphi = 1$, the separatrix is located at $v = 1$ at the phase of maximum compression, i.e., it is simply out of reach and no particles will be pushed across, no matter how large the pressure. Thus, in the limit $\gamma_\varphi \to \infty$ there should be no upper bound induced by wave breaking for the wave amplitude at all.

These observations also show that the warm-plasma approximation, which is only valid for limited compression of the plasma, cannot be used to describe ultrarelativistic waves on the verge of breaking. It has been argued that the compression on breaking is limited, so the warm-plasma approximation can be applied.[26] However, this comes down to using an approximation to justify itself, which is not necessarily correct, as the limit on compression may be (and, in fact, is) induced by employing the approximation in the first place. A more reliable method is to employ the models of K&M[20] and this paper to describe wave breaking; if the compression indeed remains small, the results will be similar to those obtained for the warm-plasma approximation, as both models are similar for small compression. Comparing the results, it is found that they are very similar for $\gamma_\varphi^2 \beta \ll 1$, so using the warm-plasma approximation is acceptable here. However, the results diverge for $\gamma_\varphi^2 \beta \gg 1$, which indicates that use of the warm-plasma approximation is not justified in that regime.

It should be noted that there are several mechanisms affecting wave propagation in a thermal plasma that are not included in our analysis, such as collisional or Landau damping. While these mechanisms will undoubtedly affect the wave amplitude after prolonged propagation, they do not necessarily cause the wave to lose its periodic structure. Therefore, they are not responsible for wave breaking and do not lower the wave-breaking limit on the amplitude, as opposed to the actual electric-field amplitude obtained using a given source. Contrary to suggestions in Refs. 24 and 40, the study of Landau damping in itself does not shed more light on the issue of warm-plasma wave breaking.

We comment on the occurrence of singularities in the plasma density at wave breaking. In the cold-plasma model by Akhiezer and Polovin,[2] wave breaking occurs when the plasma flow speed becomes equal to v_φ, leading to a singularity in the plasma density, even for finite-amplitude plasma waves. For a warm plasma, however, things are different. As long as $v_\varphi<1$, Eq. (6) guarantees that the wave breaks at $v_0<v_\varphi$, i.e., while the plasma density is still regular. For $v_\varphi=1$, on the other hand, it follows that $v_0=v_\varphi=1$, apparently allowing the plasma density to become singular on wave breaking. However, one should bear in mind that for $v_\varphi=1$, the wave-breaking amplitude is infinite, so any finite wave will not break and thus not cause the plasma density to become singular. Therefore, one concludes that the warm-plasma model discussed by K&M[20] and in this paper does not allow a finite wave to reach a singularity after all.

In conclusion, wave breaking of relativistic longitudinal waves in a warm plasma has been studied. A quantitative definition of wave breaking has been provided and compared against other definitions used in the literature. In a number of cases there, wave breaking has been equated to breakdown of the mathematical model, without adequate verification that this coincides with a physical breakdown of the wave. Having studied the fluid dynamics for a warm, relativistic plasma it has been demonstrated that only the model of K&M[20] handles this correctly. In addition, it has been investigated which models satisfy the fundamental inequality (1) by Taub,[32] and again, only the model of K&M does so for all phase speeds and wave amplitudes. This model also includes a natural correspondence between wave breaking and particle trapping (it breaks down at the exact moment that particles that are initially at the electron sound speed start to get trapped), a feature that other models investigated here lack.[24–26] It has also been shown why earlier attempts to uncover this correspondence[24,40] did not succeed. K&M's model[20] has been expanded here to derive both upper and lower limits for the electric-field amplitude E_{wb} at wave breaking, for both $\gamma_\varphi^2\beta\ll 1$ and $\gamma_\varphi^2\beta\gg 1$. It has been shown that in the latter regime $E_{wb}^2\sim\ln(\gamma_\varphi^{1/2}\beta^{1/4})/\beta^{1/2}$, which implies that for fixed β and $\gamma_\varphi\to\infty$, E_{wb} tends to infinity, so a wave with phase speed $v_\varphi=1$ will never break, regardless of its amplitude. This is particularly relevant for multi-GeV electron acceleration schemes,[6,8,10,17] as such schemes rely on the possibility of generating very large longitudinal fields in plasmas. It has also been found that several other proposed models for wave breaking in a warm plasma[24–29,40] suffer from various shortcomings that render them unsuitable for this task. Finally, a number of existing misunderstandings on the behavior of a relativistic plasma wave near breaking has been clarified.

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APPENDIX: CORRESPONDENCE BETWEEN PARTICLE AND FLUID DYNAMICS

In the literature on wave breaking, there exists some confusion on the correspondence between the macroscopic dynamics of the electron fluid and the microscopic dynamics of the individual electrons.[24,40] This correspondence will be clarified here.

In a collisionless, 1D electron plasma, the only microscopic force acting on individual electrons is the electrostatic force $-E$. This can also readily be gleaned from the 1D Vlasov equation (2), where the force factor of the $\partial f/\partial p$ term is given by $-E$ only. Taking the zeroth- and first-order moments of this equation yields the conservation equations for the plasma density n and macroscopic momentum $\langle p\rangle$,

$$\frac{\partial n}{\partial t}+\frac{\partial n\langle v\rangle}{\partial x}=0, \tag{A1}$$

$$\frac{\partial\langle p\rangle}{\partial t}+\langle v\rangle\frac{\partial\langle p\rangle}{\partial x}+\frac{1}{n}\frac{\partial P}{\partial x}=-E, \tag{A2}$$

where the relativistic pressure P is given by $P=n\langle pv\rangle-n\langle p\rangle\langle v\rangle$ and angular brackets denote the average of some microscopic quantity: $\langle h\rangle=\int h(p)f(p)dp$. There are two things that can be observed immediately. First, the macroscopic force on the fluid contains a pressure contribution that is not present in the microscopic force on individual electrons. This pressure contribution is a fictitious force, governing the exchange of energy between the collective and random parts of the total kinetic energy and acting on the mean plasma speed; however, it does not act on individual electrons. Second, since the macroscopic and microscopic forces are different, the average speed $\langle v\rangle$ and momentum $\langle p\rangle$ will not follow the orbit of any individual particle, not even the orbit of particles obeying $v=\langle v\rangle$ initially.

This behavior of $\langle v\rangle$ is of course caused by the second-order term $v\,\partial f/\partial x$ in the Vlasov equation (2) and is common for nonlinear dynamical systems. For example, consider a system for which the final speed u is a quadratic function of the initial speed u_t, $u=c_0+c_1u_t+c_2u_t^2$, and an ensemble of particles obeying $\langle u_t\rangle=0$. We then find that $\langle u\rangle=c_0+c_2\langle u_t^2\rangle$, and $\langle(u-\langle u\rangle)^2\rangle-\langle(u-c_0)^2\rangle=-c_2^2\langle u_t^2\rangle^2$. Since $u_t=0$ gives $u=c_0$, we find once again that $\langle u\rangle$ does not match the speed of a particle having $u_t=0$ initially, and that the particle velocity spread is not given by$\langle(u-c_0)^2\rangle$.

Together with an equation of state for the pressure P, Eq. (A2) can be turned into an evolution equation for the wakefield potential Ψ of the form $d^2\Psi/d\xi^2=n(\Psi)-1$. For a cold plasma one has $P=0$, of course, while for a warm plasma P depends on both the plasma density and temperature. As a result, the function $\Psi(n)$ and its inverse $n(\Psi)$ will take different forms for a cold and a warm plasma. When solving the evolution equation for Ψ, the resulting solutions will have different wave forms depending on whether a warm or a cold plasma is assumed, even if the same wave amplitude is used. This will, of course, influence the behavior of individual electrons interacting with these waves.

As stated above, the only force acting on individual plasma electrons is the electrostatic force $-E$; as a result, the potential energy V of these electrons is given by $V=-\Psi$, since $d\Psi/d\xi=-E$. Since the plasma pressure does not act on individual particles, it does not contribute separately to V, contrary to claims in Ref. 24. The single-particle Hamiltonian is thus given by $H=\gamma_p-v_{\varphi}p_p-\Psi$. The potential Ψ is determined by whether a cold or a warm plasma is used; in particular, one must not use a cold-plasma expression for Ψ if one aims to study electron dynamics in a warm plasma. The reason for this is, of course, that the wave forms in a cold and warm plasma are different, even if the wave amplitudes are chosen the same, which influences the electron dynamics. As a consequence, a bunch of test particles moving in the wakefield potential of a cold plasma does not constitute a good model for a warm plasma, contrary to claims in Refs. 24 and 40.

It is now also possible to investigate the various "mean orbits" that are used in the literature, and often mistaken for one another. In a cold plasma, the electron fluid follows a so-called "cold-fluid orbit," which coincides with the orbit of a particle having initial speed $u_t=0$. The mean speed $\langle u\rangle$ of a bunch of test particles will only follow the cold-fluid orbit if there is no velocity spread in the bunch at all. In a warm plasma, the electron fluid does not follow a cold-fluid orbit, as pointed out above. An electron that has $u_t=0$ neither follows a cold-fluid orbit (because it experiences an electric field corresponding to a warm fluid) nor the orbit of the warm electron fluid (because the plasma pressure does not act on individual particles). The mean speed of a bunch of test particles will only coincide with the speed of the warm electron fluid if their distribution functions have identical width and shape. This refutes several claims in Ref. 40.

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