Department of Physics

PH458 Topics in Quantum Physics

Friday 11th January 2013 **14:00pm -17:00pm – 3 hours**

Attempt ALL questions in Section A (40%) and TWO (out of THREE) questions from Section B (30%) and TWO (out of THREE) questions from Section C (30%)

Calculators must not be used to store text and/or formulae nor be capable of communication. Invigilators may require calculators to be reset.

Physical constants

Speed of light	$c = 3.00 \times 10^{8}$ ms^{-1}	Bohr radius	$a_0 = 5.29 \times 10^{-11}$ m
Free space permittivity	$\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}	Compton wavelength	$\lambda_C = 2.43 \times 10^{-12}$ m
Free space permeability	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}	Avogadro's number	$N_A = 6.02 \times 10^{23}$ mol^{-1}
Electrostatic constant	$k_E = 8.99 \times 10^{9}$ mF^{-1}	Boltzmann's constant	$k_B = 1.38 \times 10^{-23}$ JK^{-1}
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg	Universal gas constant	$R = 8.31$ Jmol^{-1} K^{-1}
Electron charge size	$e = 1.60 \times 10^{-19}$ C	Gravity acc. on earth	$g = 9.81$ ms^{-2}
Planck's constant	$h = 6.63 \times 10^{-34}$ J s	Gravitational constant	$G = 6.67 \times 10^{-11}$N m^{2}kg^{-2}
Wien's law constant	2.90×10^{-3} m K	Stefan-Boltzmann	$\sigma = 5.67 \times 10^{-8}$ Wm^{-2}K^{-4}
Rydberg constant	$R_\infty = 1.10 \times 10^{7}$ m^{-1}	Atomic mass unit	$u = 1.66 \times 10^{-27}$ kg

Section A

1.

a) Show that in the Coulomb gauge, Gauss' law,

$$\underline{\nabla}.\underline{E} = \frac{\rho}{\varepsilon_0}$$

can be written as Poisson's equation i.e.

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

where $\underline{E}$ is electric field, ϕ is scalar potential and ρ is charge density.

[4]

b) Show that the 1st order coherence function for an optical field with normalised frequency spectrum

$$F(\omega) = \delta(\omega - \omega_0)$$

is

$$g^{(1)}(\tau) = \exp(-i\omega_0 \tau)$$

[3]

c) Give a brief explanation of the Purcell effect.

[3]

2.

a) A signal beam of wavelength 1400nm is generated by parametric down-conversion in a non-centrosymmetric medium using a pump beam of wavelength 800nm. Calculate the wavelength of the idler beam. **[4]**

b) Explain why optical rectification is not usually possible in a centrosymmetric medium with a single optical field. **[4]**

c) Explain the term "Rabi oscillation". **[2]**

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3. Consider a quantum harmonic oscillator with raising $\hat{a}^{\dagger}$ and lowering $\hat{a}$ operators with the commutation relation $\left[\hat{a},\hat{a}^{\dagger}\right]=1$. The ground state is denoted by $|0\rangle$.

a) Using the commutation relation, show that **[4]**

$$\left(\hat{a}^{\dagger}\hat{a}\right)\hat{a}^{\dagger}=\hat{a}^{\dagger}+\left(\hat{a}^{\dagger}\right)^{2}\hat{a}\,,$$

and

$$\left(\hat{a}^{\dagger}\hat{a}\right)\left(\hat{a}^{\dagger}\right)^{2}=2\left(\hat{a}^{\dagger}\right)^{2}+\left(\hat{a}^{\dagger}\right)^{3}\hat{a}\,.$$

b) Hence, show in general that

$$\left(\hat{a}^{\dagger}\hat{a}\right)\left(\hat{a}^{\dagger}\right)^{k}=k\left(\hat{a}^{\dagger}\right)^{k}+\left(\hat{a}^{\dagger}\right)^{k+1}\hat{a}\,,$$

For k a positive integer, either directly or by mathematical induction. **[4]**

c) Using the above result, show that the unnormalised state $\left(\hat{a}^{\dagger}\right)^{k}|0\rangle$ is an eigenstate of the number operator $\hat{n}=\left(\hat{a}^{\dagger}\hat{a}\right)$. What is its eigenvalue? **[2]**

4. Consider the second order correlation function for a single mode of frequency ω,

$$g^{2}(\tau)=\frac{\left\langle\hat{E}^{-}(t)\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\hat{E}^{+}(t)\right\rangle}{\left\langle\hat{E}^{-}(t)\hat{E}^{+}(t)\right\rangle\left\langle\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\right\rangle}$$

where

$$\hat{E}^{+}(t)=K\hat{a}e^{i\omega t},\ \hat{E}^{-}(t)=K^{*}\hat{a}^{\dagger}e^{-i\omega t}$$

and K is some constant.

a) By using $\left[\hat{a},\hat{a}^{\dagger}\right]=1$, show that **[4]**

$$g^{2}(0)=\frac{\left\langle\left(\hat{a}^{\dagger}\hat{a}\right)^{2}\right\rangle}{\left\langle\hat{a}^{\dagger}\hat{a}\right\rangle^{2}}-\frac{1}{\left\langle\hat{a}^{\dagger}\hat{a}\right\rangle}$$

b) Calculate $g^{2}(0)$ for the state **[4]**

$$|\psi\rangle=\cos\theta|0\rangle+\sin\theta|2\rangle$$

where $0<\theta<\frac{\pi}{2}$.

c) For what values of θ does $g^{2}(0)$ take on non-classical values? **[2]**

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Section B

5. a) Doppler broadened light with central wavelength $\lambda_0 = 780\ \text{nm}$ from a gas of Rb atoms is passed through a Young's (two-slit) interferometer. At a position on the screen where the path difference to the slits is 5cm, the fringe visibility is 0.9. Calculate the temperature of the gas, T, if the Doppler linewidth is defined as

$$\delta_D = \omega_0 \sqrt{\frac{k_B T}{mc^2}}$$

Where m is the mass of Rb (85.5 atomic mass units) and ω_0 is the central frequency of the emitted light.

[7]

b) The electric field envelope of an optical pulse is given by

$$E(t) = E_0 \exp\left(-\frac{t^2}{2\tau^2}\right)$$

Where E_0 is the peak electric field and τ characterises the pulse duration. If the pulse has peak intensity $I_0 = 10^8\ \text{Wm}^{-2}$ and is resonant with a transition between two states of an atomic gas, calculate the value of τ required for self-induced transparency to occur. Assume the dipole moment of the atomic transition is $\mu = 10^{-29}$ *C m*.

Use the definite integral: $\int_{-\infty}^{\infty} \exp(-ax^2)dx = \sqrt{\frac{\pi}{a}}$ where a is a real, positive constant.

[8]

6. a) The wave equation for an electric field in a dielectric medium is

$$\nabla^2 \underline{E}(t) - \frac{1}{c^2}\frac{\partial^2 \underline{E}(t)}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \underline{P}(t)}{\partial t^2}$$

where $\underline{P}$ is the polarisation. Show that in a hypothetical medium where the dominant nonlinearity is fifth-order, the component of the electric field with angular frequency ω will satisfy

$$\left(\nabla^2 + \frac{\omega^2}{c'^2}\right)\underline{E}(\omega) = 0$$

where

$$c' = \frac{c}{\sqrt{\left(1 + \chi^{(1)} + \chi^{(5)}|E(\omega)|^4\right)}}$$

$\chi^{(1)}$ and $\chi^{(5)}$ are the linear and fifth-order susceptibilities respectively.

[10]

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b) Show that the refractive index, n, of a medium with fifth-order nonlinearity can be written as

$$n = n_0 + \Delta n$$

Where $n_0 = \sqrt{1+\chi^{(1)}}$ and $\Delta n = \frac{\chi^{(5)}}{2n_0}|E(\omega)|^4$ when $\chi^{(5)}|E(\omega)|^4 << n_0{}^2$.

[5]

7. The rate equation for the population of the upper level of a nondegenerate two-level atomic system, $N_2(t)$, interacting with an electromagnetic field with spectral energy density, $u(\omega)$, is

$$\frac{dN_2}{dt} = -A\ N_2 + Bu(\omega)(N_1 - N_2)$$

where $N_1(t)$ is the population of the lower level and A and B are Einstein coefficients.

a) Show that the rate equation for the population of the lower level, $N_1(t)$, can be written as

$$\frac{dN_1}{dt} = -(A + 2Bu(\omega))N_1 + N(A + Bu(\omega))$$

Where $N = N_1(t) + N_2(t)$ is the total number of atoms.

[5]

b) Using a trial solution of the form

$$N_1(t) = C\exp(-(A + 2Bu(\omega))t) + D$$

Where C and D are constants, show that a solution to the rate equation for $N_1(t)$ in 7a) is

$$N_1(t) = \frac{N}{A + 2Bu(\omega)}\left[Bu(\omega)\exp(-(A + 2Bu(\omega))t) + A + Bu(\omega)\right]$$

if all the atoms are in the lower level initially i.e. $N_1(t=0) = N$.

[6]

c) Show that in the limit of a strong optical field the steady-state lower-level population approaches $N_1 = \frac{N}{2}$ and that in the limit of a weak optical field the steady-state lower-level population approaches $N_1 = N$. **[4]**

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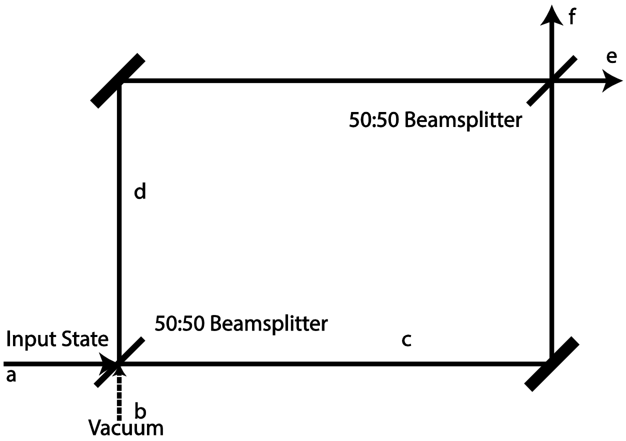
Section C

8. Consider the balanced Mach-Zehnder interferometer with arms of equal length (i.e. no relative phase-shift), and we ignore any phase shift due to reflection from the fully reflective mirrors. We put an input state into mode a ,and nothing (vacuum) into mode b .



The input-output relations for the two beam splitters are

$$\hat{a}^\dagger \to \left(\hat{c}^\dagger + i\hat{d}^\dagger\right)/\sqrt{2},\ \hat{b}^\dagger \to \left(i\hat{c}^\dagger + \hat{d}^\dagger\right)/\sqrt{2}$$
$$\hat{c}^\dagger \to \left(i\hat{e}^\dagger + \hat{f}^\dagger\right)/\sqrt{2},\ \hat{d}^\dagger \to \left(\hat{e}^\dagger + i\hat{f}^\dagger\right)/\sqrt{2}$$

respectively. Ideal photon number detectors are placed at the outputs of e and f .

a) For an input consisting of the coherent state $\left|\alpha\right\rangle_a = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left|n\right\rangle_a$, show that it can be expressed as $\left|\alpha\right\rangle_a = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} \left|0\right\rangle$ **[3]**

b) Show that the intermediate state (between first and second beam splitters) is $\left|\frac{\alpha}{\sqrt{2}}\right\rangle_c \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_d$ i.e. a product state of two coherent states of equal magnitude but with a relative phase of i in each arm of the interferometer. **[5]**

c) After the second beam splitter (at e and f), what is the output state? What will each of the photo detectors measure? **[5]**

d) If we were instead to put in a number state $\left|n\right\rangle_a$ at the input a , how does it evolve and what would each of the photo detectors at e and f measure? **[2]**

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9. Consider a two-level atom (ground state $|g\rangle$ and excited state $|e\rangle$) resonantly coupled to a single mode of a cavity (annihilation operator $\hat{a}$) with the Hamiltonian given by the Jaynes-Cummings model

$$\hat{H}_{JC} = \frac{\hbar\omega(1+\hat{\sigma}_z)}{2} + \hbar\omega\hat{a}^\dagger\hat{a} - \hbar\gamma\left(\hat{\sigma}_-\hat{a}^\dagger + \hat{\sigma}_+\hat{a}\right)$$

Where $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\hat{\sigma}_- = |g\rangle\langle e|$, and $\hat{\sigma}_+ = |e\rangle\langle g|$, and γ is the coupling rate and we have ignored the vacuum energy.

a) Show that the combined entangled states of the atom and cavity given by $|\psi_n^\pm\rangle = \frac{1}{\sqrt{2}}\left(|g,n\rangle \mp |e,n-1\rangle\right)$ are eigenstates of the Hamiltonian with eigenenergies $E_n^\pm = n\hbar\omega \pm \hbar\gamma\sqrt{n}$ respectively. **[6]**

b) Initially, the atom is in its ground state $|g\rangle$ and the cavity mode is in the state $\frac{1}{\sqrt{2}}\left(|1\rangle + |4\rangle\right)$. Show that the initial state of the atom-cavity system is in an equal superposition of energy eigenstates of $\hat{H}_{JC}$ **[5]**

$$|\varphi(0)\rangle = \frac{1}{2}\left(|\psi_1^-\rangle + |\psi_1^+\rangle + |\psi_4^-\rangle + |\psi_4^+\rangle\right)$$

c) An initial state in a superposition $|\varphi(0)\rangle = \sum_j c_j |E_j\rangle$ of energy eigenstates $|E_j\rangle$ with energy eigenvalues E_j evolves into $|\varphi(t)\rangle = \sum_j e^{\frac{-iE_j t}{\hbar}} c_j |E_j\rangle$. The probability of the atom to be found in the ground state is

$$\mathrm{Pr}_g(t) = \frac{1}{2}\left[\cos^2(\gamma t) + \cos^2(2\gamma t)\right].$$

Sketch the probability of the atom to be in its ground state as a function of time in units of γt, $0 \le \gamma t \le \pi$. When will the atom next return to its ground state exactly? **[4]**

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10. The characteristic function of a state ρ is given by

$$\chi(\eta) = Tr\left[\rho e^{\eta \hat{a}^\dagger - \eta^* \hat{a}}\right]$$

a) Show that the characteristic function of the vacuum is given by $\chi(\eta) = e^{\frac{-|\eta|^2}{2}}$ **[4]**

b) The Wigner distribution $W(\alpha)$ is the Fourier transform of the characteristic function,

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\eta\ e^{\eta^*\alpha - \eta\alpha^*}\chi(\eta)$$

where the integral is taken over all complex values of η. Show that the Wigner distribution of the vacuum is a Gaussian centred on the origin, $W(\alpha) = \frac{2}{\pi}e^{-2|\alpha|^2}$. Hint: Split the complex integral into real and imaginary parts and use the identity, $\int_{-\infty}^{\infty} dt\ e^{-at^2}e^{ibt} = \sqrt{\frac{\pi}{a}}e^{\frac{-b^2}{4a}}$ for $a > 0$, b real. **[6]**

c) Show that the single mode Hamiltonian $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ can be expressed in the Weyl symmetric product form **[2]**

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$$

d) Denoting the Weyl symmetric product as $S\left[\hat{a}^n\left(\hat{a}^\dagger\right)^m\right]$, the expectation value of an operator expressed as $\hat{X} = \sum_{n=0,m=0}^{\infty}\chi_{nm}^X\ S\left[\hat{a}^n\left(\hat{a}^\dagger\right)^m\right]$ can be evaluated as

$$\left\langle\hat{X}\right\rangle = \int d^2\alpha\ \chi^X(\alpha)W(\alpha)$$

Where we have defined $\chi^X(\alpha) = \sum_{n=0,m=0}^{\infty}\chi_{nm}^X\alpha^m\left(\alpha^*\right)^n$, (same χ_{nm}^X as in $\hat{X}$). In this way, determine the single mode vacuum energy $\left\langle\hat{H}\right\rangle$. Hint: Use the polar decomposition and the integral identity $\int_0^{\infty} dr\ r^3 e^{-ar^2} = \frac{1}{2^{a+1}}$ where $a > 0$ **[3]**

END OF PAPER

(Dr.D. Oi and Dr.G. Robb**)**