1st Year Report

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1 Motivation

Using spontaneous parametric down conversion (SPDC) the orbital angular momentum (OAM) states $|\psi^{\pm}\rangle = \sum_{l=-L}^{L} c_l |l, \pm l\rangle$ are relatively easy to create. The anti-correlated OAM state occurs readily in SPDC due to conservation of angular momentum and the entanglement itself arises from the transverse phase-matching conditions [1, §8][6]. The other entangled state with correlated angular momenta can be created from the anti-correlated OAM entangled state by a dove prism. These OAM entangled states are useful because they allow for the study of two-particle entanglement in higher dimensional state spaces than electron spin and photon polarisation systems have allowed for in the past. These two states provide a possible extension of the Bell states to higher dimensional spaces.

Our study has primarily a photonic implementation in mind but there is no reason to restrict theoretical investigation into these entangled states to bosons only as there is considerable interest in electron vortices carrying orbital angular momentum. We consider both bosons and fermions in the above entangled state[1].

The starting point was to ask how these two states upon propagation through a beam splitter. This starting point was motivated by a view towards optical quantum computing and the fact that a beam splitter is a device that can couple optical modes without the low probabilities that come with using non-linear effects [5].

We denote the input modes by $\hat{a}^{\dagger}_{p,l}$ and the output modes by $\hat{b}^{\dagger}_{p,l}$ (see figure 1) where p labels the port of the beam splitter and l labels the angular momentum of the particle. The beam splitter couples the input and output modes according to

$$\begin{aligned} \hat{a}^{\dagger}_{1,l} &= t\hat{b}^{\dagger}_{1,l} + r\hat{b}^{\dagger}_{2,l} \\ \hat{a}^{\dagger}_{2,l} &= r\hat{b}^{\dagger}_{1,l} + t\hat{b}^{\dagger}_{2,l}. \end{aligned} \tag{1}$$

The requirement that input and output modes satisfy the usual commutation (anticommutation) relations for bosonic (ferminoic) creation and annihilation operators impose the relations

$$\begin{aligned} |t|^2 + |r|^2 &= 1 \\ r^*t + rt^* &= 0. \end{aligned}$$

To satisfy these relations we use the convention $t = |t|$ and $r = \mathrm{i}|r|$.

2 Methods: exchange symmetry considerations

Investigation is guided primarily by exchange symmetry considerations of the two particle state[2]. It is well known that a system of two indistinguishable particles must either be symmetric or antisymmetric under particle exchange and that this symmetry of a two particle state under particle exchange does not change [7]. This is because to respect the indistinguishability of identical particles any Hamiltonian on two identical particles must be invariant under the exchange of these two particles. Hence the Hamiltonian commutes with the particle exchange operator and the symmetry of the state under

[1] I cite the talks given at the Spin-Orbit Interaction for Light and Matter Waves conference which was organised at the Max-Planck-Institute for Complex Systems in Dresden in April 2013 to back up this statement

[2] We consider only two particle systems for the moment however there is potential to apply the tools of exchange symmetry to entangled states of more than two particles. However OAM entangled states of 3 particles are hard to produce HAS BEEN DONE?

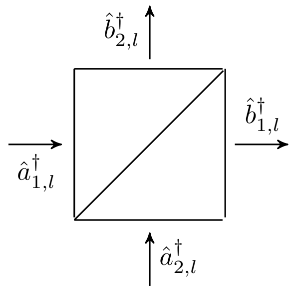

Figure 1: The labels of the two input ports and the two output ports.

exchange is a constant of motion.

Now consider the exchange of two particles in the setup under consideration. In second quantised notation a two particle state is labeled by the product[3] of two creation operators $\hat{a}^\dagger_{p,l}\hat{a}^\dagger_{q,j}$. The particles are labeled by which port they are in $p, q \in \{1, 2\}$ and their orbital angular momentum $l, j \in \mathbb{Z}$. To achieve the exchange of two particles one must exchange both these numbers between the two creation operators. To this end we can define exchange operators $\hat{\mathsf{P}}$ and $\hat{\mathsf{L}}$ that exchange the port and OAM numbers respectively. Thus these two operators can then be used to define the particle exchange operator

$$\hat{\mathsf{P}}\hat{\mathsf{L}} = \hat{\mathsf{L}}\hat{\mathsf{P}} =: \hat{\mathsf{E}}. \tag{2}$$

The relevant phase (0 or π) due to particle exchange is introduced by the commutation and anticommutation relations

$$[\hat{a}^\dagger_{p,l}, \hat{a}^\dagger_{q,j}] = \{\hat{a}^\dagger_{p,l}, \hat{a}^\dagger_{q,j}\} = 0$$

for the boson and fermion cases respectively. For states that have at most one particle in a port of the beam splitter[4], such as the input states that we are investigating, it is also possible to treat these operators in bra-ket notation however for the case of 2 particle per port states[5] second quantised notation is far more convenient. The main results of exchange symmetry considerations can be obtained from considering the input states only which can be represented in bra-ket notation without the inherent clutter of second quantised notation due to the fact that all the important symbols are subscripts. To this end let us choose the convention that the first label in the ket always describes the OAM in port mode 1 and the second label always describes the OAM in port 2. Given this convention the effect of $\hat{\mathsf{L}}$ is simply to exchange the two OAM labels,

$$\hat{\mathsf{L}}\,|l, j\rangle = |j, l\rangle\,.$$

Now we know that the exchange of two bosons results in the same state and the exchange of the two fermions results in a sign change so we can write that $\hat{\mathsf{E}} = (-1)^T\hat{1}$ where the variable T (for 'type') is 0 for bosons and 1 for fermions and $\hat{1}$ is the identity operator. Then we can deduce that the effect of $\hat{\mathsf{P}}$ is to exchange the OAM labels and introduce the relevant phase

$$\hat{\mathsf{P}}\,|l, j\rangle = (-1)^T\,|j, l\rangle$$

as the subsequent application of $\hat{\mathsf{L}}$ and $\hat{\mathsf{P}}$ must result in $\hat{\mathsf{E}} = (-1)^T\hat{1}$. While the two operators $\hat{\mathsf{L}}$ and $\hat{\mathsf{P}}$ may not be physical they are well defined and whether a state is symmetric or antisymmetric[6] under either of these does have physical consequences. The operators $\hat{\mathsf{P}}$ and $\hat{\mathsf{L}}$ commute with the application of the beam splitter transformation. Thus we can define the concepts of port (anti)symmetry and OAM (anti)symmetry and conclude that they remain unchanged under the beam splitter transformations. Any exchange symmetry argument for the input states, which have a relatively simple form, remains valid for the output states, which may be more complicated in form, for example due to the introduction

[3] and by the principle of superposition any linear combination of such products

[4] Because we consider two particles in a two output port device this means that the state is necessarily described over two ports so we refer to it as a two port state

[5] Again due to considering two particles in two ports this is then necessarily a state over one port and we hence refer to it as a one port state.

[6] States that are symmetric under $\hat{\mathsf{P}}$ we refer to as $\hat{\mathsf{P}}$-symmetric and states antisymmetric under $\hat{\mathsf{P}}$ we refer to as $\hat{\mathsf{P}}$-antisymmetric and likewise for $\hat{\mathsf{L}}$.

of two particle per port states.

It is useful to think of a more general form of writing the input states $|\psi^{\pm}\rangle$ and observing the special cases they correspond to. This will allow for their classification of these states based on their properties under exchange.

$$\sum_{l,j=-L}^{L} c_{lj} |l,j\rangle = \sum_{l=-L}^{L} c_{ll} |l,l\rangle + \sum_{l=-L}^{L} \sum_{j=-L}^{l-1} \left(c_{lj} |l,j\rangle + c_{jl} |j,l\rangle \right). \tag{3}$$

We restrict ourselves to an arbitrary but finite $(2L+1)$ dimensional space so that the coefficients c_{ij} may be viewed as a finite matrix. We can consider the state to be described by the $(2L+1)\times(2L+1)$ matrix of coefficients c_{lj}. The argument of the double summation in the second term can be rewritten as

$$\frac{c_{lj}+c_{jl}}{2} \left(|l,j\rangle + |j,l\rangle \right) + \frac{c_{lj}-c_{jl}}{2} \left(|l,j\rangle - |j,l\rangle \right).$$

Thus we see that there are three distinct classes of states present in (3).

1. states described by the diagonal (a necessarily symmetric part) of the matrix $d_l := c_{ll}$
2. states described by the symmetric component of the off-diagonal part of the matrix $c_{(lj)} := \frac{c_{lj}+c_{jl}}{2}$
3. states described by the anti-symmetric component of the off-diagonal part of the matrix $c_{[lj]} := \frac{c_{lj}-c_{jl}}{2}$.

The state corresponding to each of the above types of coefficients have similar symmetry properties to the symmetry properties of the coefficients. Respectively the corresponding states are

1. OAM-degenerate and hence $\hat{\mathsf{L}}$-symmetric states $|l,l\rangle$
2. non-OAM-degenerate but $\hat{\mathsf{L}}$-symmetric states $|l,j\rangle + |j,l\rangle$
3. $\hat{\mathsf{L}}$-antisymmetric states $|l,j\rangle - |j,l\rangle$

Thus the symmetric and anti-symmetric parts of the matrix c_{lj} correspond to $\hat{\mathsf{L}}$-symmetric and $\hat{\mathsf{L}}$-antisymmetric parts of the state. This statement is true for bosons as well as fermions given that we use the convention outlined earlier[7]. By tuning the ratio between these components in the matrix c_{lj} we can tune the ratio of $\hat{\mathsf{L}}$-symmetric and $\hat{\mathsf{L}}$-antisymmetric parts of the state and hence also the $\hat{\mathsf{P}}$-symmetric and $\hat{\mathsf{L}}$-antisymmetric parts. This can be done by introducing a relative phase between c_{lj} and c_{jl}.

3 Some consequences of exchange symmetry applied to $|\psi^{\pm}\rangle$

To understand the physical significance of $\hat{\mathsf{P}}$-(anti)symmetry consider a state of two particles in one port. This state is degenerate in port number and hence $\hat{\mathsf{P}}$-symmetric. That means that the amplitude distribution of OAM must be $\hat{\mathsf{L}}$-symmetric for bosons and $\hat{\mathsf{L}}$-antisymmetric for fermions. In both cases the OAM probability distribution is symmetric due to the absolute value involved in going from amplitudes to probabilities.

Now consider the input state $|\psi^{+}\rangle$. This is an OAM-degenerate state and hence $\hat{\mathsf{L}}$-symmetric. An equivalent approach is to say that it corresponds to the diagonal state $c_{lj} = c_l \delta_{lj}$. Therefore fermions in the $|\psi^{+}\rangle$ input state cannot scatter into the same output port and must scatter to different ports independently of the value of t^2+r^2. This value of t^2+r^2 arises in Hong-Ou-Mandel (HOM) scattering as the coefficient of the term that describes bosons in different ports [3] but it does not appear for the corresponding fermionic calculation. Bosons scatter strictly into the same port only when the beam

[7] It may appear that it depends on the notational convention whether the state is symmetric or antisymmetric under a given exchange operator. This is not the case but we do have a freedom to write the state in two forms. We can either write it in $\hat{\mathsf{L}}$-symmetrised and $\hat{\mathsf{L}}$-antisymmetrised form independently of whether we are speaking of fermions or boson and then the $\hat{\mathsf{P}}$ symmetry will be dependent on whether the state is fermionic or bosonic or we can write it in $\hat{\mathsf{P}}$-symmetrised and $\hat{\mathsf{P}}$-antisymmetrised form independently of bosonic or fermionic nature and the $\hat{\mathsf{L}}$-symmetry will conform according to $\hat{\mathsf{P}}\hat{\mathsf{L}} = (-1)^T \hat{1}$. Bra-ket notation only lends itself readily to the former but in second quantised notation the former corresponds to OAM-ordered form and the latter corresponds to port-ordered form.

splitter is a 50:50 one.

Using the beam splitter relations (1) it can be computed that for bosons the probabilities of scattering into the same port and into different ports are the same as in the case of HOM interference

$$
\begin{aligned}
P\,(\text{scatter into the same port}) &= 2|t|^2|r|^2 \\
P\,(\text{scatter into different ports}) &= \left(|t|^2 - |r|^2\right)^2.
\end{aligned}
$$

In fact the HOM interference is a special case of $|\psi^+\rangle$ scattering for $c_0 = 1, c_{l\neq 0} = 0$. This suggests that the deviation of $|t|^2 - |r|^2$ from zero controls to what extent two port output states can be $\hat{\mathsf{P}}$-symmetric. In the special case of $|t|^2 - |r|^2 = 0$ a state of one boson in each port is purely $\hat{\mathsf{P}}$-antisymmetric.

Now consider the input state $|\psi^-\rangle$ with the $l = 0$ term removed[8]

$$
|\psi^-\rangle = \sum_{l=-L, l\neq 0}^{L} c_{l-l}\,|l,-l\rangle = \sum_{l,j=-L}^{L} c_l \delta_{-l,j}\,|l,j\rangle\,.
$$

By the beam splitter transformation (1) it emerges that the output state depends on the relation of c_l to c_{-l} or in the more general notation c_{l-l} to c_{-ll}. We introduce a relative phase between these terms according to

$$
c_{l-l} = \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(l)\phi_R} c_{-ll}. \tag{4}
$$

In this relation it is implicit that we assume the special case $|c_{lj}| = |c_{jl}|$. The the symmetric and antisymmetric components of this antidiagonal tensor then become

$$
\begin{aligned}
c_{(l-l)} &= c_{l-l}\mathrm{e}^{-\mathrm{i}\frac{\phi_R}{2}} \cos\left(\frac{\phi_R}{2}\right) \\
c_{[l-l]} &= c_{l-l}\mathrm{e}^{-\mathrm{i}\frac{\phi_R}{2}}\,\mathrm{i}\sin\left(\frac{\phi_R}{2}\right).
\end{aligned}
$$

Then the $\hat{\mathsf{L}}$-symmetric and $\hat{\mathsf{L}}$-antisymmetric components take the very familiar form of a qubit

$$
\sum_{|l|=1}^{L} c_l\,|l-l\rangle = \mathrm{e}^{-\mathrm{i}\frac{\phi_R}{2}} \left\{ \cos\left(\frac{\phi_R}{2}\right) \sum_{l=1}^{L} c_{l-l}\left(|l,-l\rangle + |-l,l\rangle\right) + \mathrm{i}\sin\left(\frac{\phi_R}{2}\right) \sum_{l=1}^{L} c_{l-l}\left(|l,-l\rangle - |-l,l\rangle\right) \right\}.
$$

Again by the beam splitter relation (1) for the 50:50 beam splitter it can be shown that

$$
\begin{aligned}
P\,(\text{scatter into the same port}) &= \cos^2\left(\frac{\phi_R}{2}\right) \\
P\,(\text{scatter into different ports}) &= \sin^2\left(\frac{\phi_R}{2}\right)
\end{aligned}
$$

for bosons and exactly the opposite is true for fermions, again showing that for a 50:50 beam splitter the two-port output state is strictly $\hat{\mathsf{P}}$-antisymmetric and the one-port output state is $\hat{\mathsf{P}}$-symmetric. The 50:50 beam splitter acts as an $\hat{\mathsf{L}}$-symmetry sorter.

The notion of label exchange operators allows us to say quite a lot about the angular dependence of the output states. We do this by defining the angle exchange operator $\hat{\mathsf{A}}$. The action of this operator is defined in the angle basis as it exchanges angle numbers. We have then the relation that

$$
\hat{\mathsf{P}}\hat{\mathsf{A}} = \hat{\mathsf{P}}\hat{\mathsf{L}} = \hat{\mathsf{E}}.
$$

This means that the state has the same exchange symmetry in angle as it does in OAM. Consider again a 50:50 beam splitter and the input state $|\psi^-\rangle = \sum_{l=-L,l\neq 0}^{L} c_l\,|l,-l\rangle$. The two particle per port output states are $\hat{\mathsf{P}}$-symmetric and so must be $\hat{\mathsf{A}}$-symmetric for bosons and $\hat{\mathsf{A}}$-antisymmetric for fermions so both must give angle probability distributions that are symmetric under the exchange of the two angles. Given that for a 50:50 beam splitter the two port output state is $\hat{\mathsf{P}}$-antisymmetric it must also be $\hat{\mathsf{A}}$-antisymmetric for bosons and $\hat{\mathsf{A}}$-symmetric for fermions hence the angle probability distributions

[8]The term $c_0\,|0,0\rangle$ is an OAM degenerate one and we want its behaviour to be absent for now so that we may concentrate on the off diagonal terms. This term really belongs to the diagonal state $c_{lj} = c_l\delta_{l,j}$ in behaviour.

must again by $\hat{\mathsf{A}}$-symmetric in both cases (see figure 2)[9]. However when it is no longer the case that the transmission and reflection coefficients have the same magnitude the two port output state is no longer restricted to having a well defined symmetry under $\hat{\mathsf{P}}$ meaning that it can be a superposition of $\hat{\mathsf{P}}$-symmetric and a $\hat{\mathsf{P}}$-antisymmetric component. Hence the resulting angle probability distributions no longer have a well defined symmetry under angle exchange. The $\hat{\mathsf{P}}$-symmetry of a two particle per port state however cannot be changed so the distributions will always be symmetric (see figure 3).

4 Next steps

The next immediate possible directions in investigating this problem are the following.

- The relative phase (4) needs some consideration as it is required to be dependent on the sign of l but not on its magnitude. The experimental implementation of it does not seem to be a trivial one. Perhaps the extension of the results to an l-dependent relative phase might be required to see if sufficiently many effects of the relative phase survive if an l-dependent relative phase is readily available in an experimental setup.
- As the main principles guiding the research are those of symmetries under particle exchange a sufficient framework is in place to investigate some properties of three particle entangled states

$$\sum_{l_1,l_2,l_3=-L}^{L} c_{l_1,l_2,l_3} \left| l_1, l_2, l_3 \right\rangle .$$

  In this extention one thing to note is that the coefficients c_{lj} of the two particle system can be viewed as matrices but they are more specifically rank-2 tensors. This follows straight forwardly from the fact that changing basis by local unitaries $|l\rangle = \sum_{l'} U_{ll'} |l'\rangle$ results in $c_{l'j'} = \sum_{l,j} U_{ll'} U_{jj'} c_{lj}$. The two of the three main ingredients to make this extension have been explored already in the two particle case. These are the symmetries of rank-2 tensors and exchange symmetries of the two particle state. In the three particle state the tensors of concern will be rank-3 and the third ingredient will be the symmetric group S_3, the group of permutations of 3 objects which will govern the relation of the symmetries of rank-3 tensors to the symmetries under of three particle states under exchanging particles. However three particle OAM entangled states are not as readily (if at all) available experimentally as two particle entangled states.
- Consideration needs to be given to what this classification of entangled states according to $\hat{\mathsf{P}}, \hat{\mathsf{L}}$ and $\hat{\mathsf{A}}$ symmetry implies for some protocols such as the KLM protocol [4] and teleportation [2]. Teleportation should be an interesting case because the role of exchange symmetry plays an important role in the teleportation of a two dimensional state. The question is whether this extends in a clean way to a higher dimensional state space.

References

[1] L. Allen, Stephen M. Barnett, and Miles J. Padgett. *Optical Angular Momentum*. Institute of Physics Publishing, 2003.

[2] Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter, and Anton Zeilinger. Experimental quantum teleportation. *Nature*, 390, 1997.

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[5] Pieter Kok, W.J. Munro, Kae Nemoto, T.C. Ralph, Jonathan P. Dowling, and G.J. Milburn. Linear optical quantum computing with photonic qubits. *Reviews of Modern Physics*, 79, 2007.

[9] When consulting the figures note that all angle probability distributions resulting from the antidiagonal state $|\psi^-\rangle$ are a functions of $\Theta - \Theta'$ because they are governed by the Fourier transform of c_{lj} and the δ_{-lj} structure of it ensures this condition.

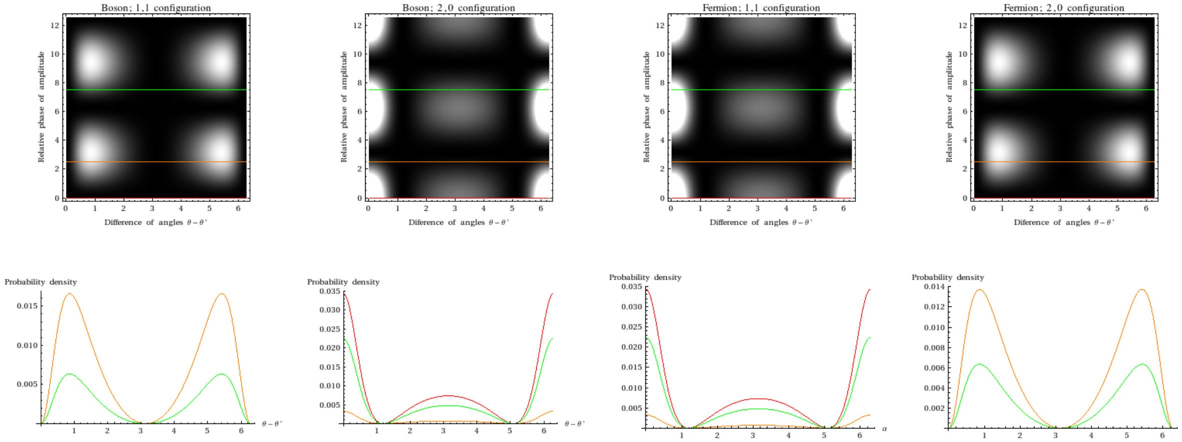

Figure 2: Plot of the angle probability distributions for $|t|^2 = |r|^2 = \frac{1}{2}$ and with $|c_{lj}| = \mathrm{e}^{-|l|}\delta_{-lj}$. The phase dependence is given by $c_{lj} = \mathrm{e}^{-|l|}\delta_{-lj}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(l-j)\frac{\phi_R}{2}}$ to satisfy the relation (4). The left two column of figures describes the functions for bosons and the right two columns describe the functions for fermions. Density plots are functions of relative phase ϕ_R on the vertical axis and functions of $\Theta-\Theta'$ on the horizontal axis. Each lineplot describes a cross section of the 2d plot of the corresponding position. The coloured line accross the 2d plot corresponds to the cross section that the correspondingly coloured lineplot describes.

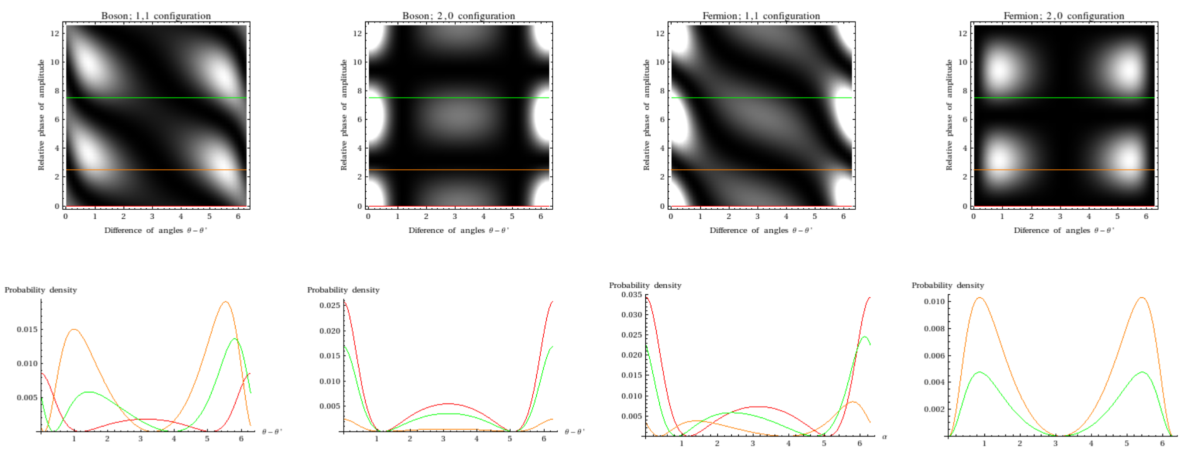

Figure 3: Plot of the angle probability distributions for $|t|^2 = \frac{1}{4}, |r|^2 = \frac{3}{4}$ and with $c_{lj} = \mathrm{e}^{-|l|}\delta_{-lj}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(l-j)\frac{\phi_R}{2}}$. The system according to which the plots are laid out is identical to that in figure 2.

[6] Filippo M. Miatto, Alison M. Yao, and Stephen M. Barnett. Full characterization of the quantum spiral bandwidth of entangled biphotons. *Physical Review A*, 83, 2011.

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